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The compression of a thin strip of material in a superplasticity state☆

I.A. Kiiko

Moscow, Russia

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ABSTRACT

The viscoplastic flow of a thin strip of material in a superplasticity state between rigid, converging parallel planes (an analogue of Prandtl's problem) is investigated. An analytical quadrature solution of the problem is constructed, asymptotically precise in the same sense as Prandtl's solution. Special cases are considered where the solution (including an approximate solution) is written out fully. The effects of superplasticity are determined.

Modern machining technologies make wide use of the superplastic deformation effect, so the development of mathematical models of such processes is an extremely urgent problem. It is fundamental that the model should reflect the conditions characterizing the transition of the material to the superplastic state and its behaviour in that state: a comparatively narrow temperature range, grain size and strain rate limitations and the strong dependence of the properties on these parameters. However, an analysis of publications addressing these problems shows[1] that the vast majority of them give no mathematically correct formulation or analysis of the initial boundary-value problems modelling a particular production process.

1. Formulation of the problem

We will consider a viscoplastic material occupying a region in the form of a rectangle $|x| \le l$, $|y| \le h$. The sides $y = \pm h$ converge at a velocity $\mp \upsilon_0$, so that the layer thickness $h = h(t)$ is a known function of time (from the incompressibility condition, the dimension $l = l(t)$ is also known). In view of symmetry, it is sufficient to consider the flow in the right-hand quadrant $0 \le x \le l$, $0 \le y \le h$.

The symbols u and υ will denote the vector components of the flow velocity in an Euler description of the motion; accordingly, we have the strain rate tensor (deviator) with the components

$$\upsilon_{xx} = \frac{\partial u}{\partial x}, \quad \upsilon_{yy} = \frac{\partial \upsilon}{\partial y}, \quad \upsilon_{xy} = \upsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right) \tag{1.1}$$

and its intensity (given the incompressibility condition $\partial u/\partial x + \partial \upsilon/\partial y = 0$)

$$\upsilon_u = 2\sqrt{\upsilon_{xx}^2 + \upsilon_{xy}^2} \tag{1.2}$$

The symbol σ_{ij} will denote the true stress tensor components, and $S_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ $(i, j = x, y)$ will denote the components of its deviator; under conditions of plane strain, $\sigma_{zz} = \sigma = (\sigma_{xx} + \sigma_{yy})/2$. We will assume that the intensity of the stresses σ_u is a known function of the process parameters.

The stresses satisfy the equilibrium equations

$$\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} = \frac{\partial \sigma}{\partial x}, \quad \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} = \frac{\partial \sigma}{\partial y} \tag{1.3}$$

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E-mail address: krok@rambler.ru.

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and are related to the process kinematics by Saint-Venant's law of plastic flow

$$S_{ij} = \frac{2\sigma_u}{3\upsilon_u}\upsilon_{ij}, \quad i,j = x,y \tag{1.4}$$

Before formulating the problem, we will write down the conditions of the superplastic state (SP state):

(1) The temperature lies in a specified range: $T_1 \le T \le T_2$; the extreme values T_1 and T_2 differ very little and the temperature can therefore be replaced by a certain average value (note that the vast majority of production processes with materials in the SP state are conducted under isothermal conditions);
(2) the grain size has an upper limit: $d \le d_m$.
(3) the strain rate υ_u also has an upper limit: $\upsilon_u \le \upsilon_{\max}$.

In the law of flow (1.4), σ_u depends on the parameters mentioned, and the grain size is given by the evolutionary equation

$$\dot{d} = \psi(d, \upsilon_u, T) \tag{1.5}$$

An example of this equation for isothermal conditions is given below.

Under fairly general initial and boundary conditions, constraints 1 to 3 single out, in the entire flow region, subregions occupied by material in the SP state; the boundaries of the subregions are not known in advance and are determined when solving the problem. It is clear that, in such a formulation, it is practically impossible to obtain an analytical solution even of a relatively simple problem, including the problem examined here. We will therefore adopt the following hypotheses:

1°. The process takes place, as in most technologies, at a certain constant temperature in the range (T_1, T_2).
2°. The condition $d < d_m$ is satisfied; the dependence of σ_u on d at constant strain rate and temperature (this is important) is described by the well-known Hall–Petch empirical equation

$$\sigma_u = \sigma_0 + K/\sqrt{d} \tag{1.6}$$

in which σ_0 and K are constants of the material. It can be assumed that Eq. (1.6) can be generaliged (in the isothermal case) as follows:

$$\sigma_u = \sigma_0(\upsilon_u) + \sigma_1(\upsilon_u, d) \tag{1.7}$$

but the possible form of the functions $\sigma_0(\upsilon_u)$ and $\sigma_1(\upsilon_u, d)$ is not conclusively ascertained. It is well known, however, that the dependence of σ_u on d is monotonic.[3]
3°. Outside the SP state region, the material is viscoplastic: $\sigma_u = \sigma_u(\upsilon_u)$. In what follows, in order to obtain relatively clear analytical results, we will assume that, outside the SP state region, the material is ideally plastic with a yield point σ_0.
The following assumptions will be made on the basis of the characteristic features of Prandtl's solution:[2] the strain and stress rate deviators do not depend on the x coordinate; the strain rate intensity υ_u in the region $0 \le h \le h$ varies from a certain finite magnitude to infinity–this feature is most important.
4°. We will assume that the velocity component υ and and the strain rate deviator are functions of only the y coordinate and time (as a parameter), from which it follows that the stress deviator is also a function of only y and t.

As can be seen, the mathematical model of the problem is a complex constrained system of non-linear equations; it is natural to solve it by successive approximations. In a first approximation, in the SP state region, we assume that $\sigma_u = F_1(\upsilon_u) = \sigma_0 F(\upsilon_u/\upsilon_{\max})$ and that this region is defined by the condition $\upsilon_u = \upsilon_{\max}$. From the last hypothesis it follows that the boundary of the SP state region will be the line $y = \eta(t)$. A further consideration in favour of the successive approximation method is the well known characteristic of the strain hardening function $\sigma_u = \sigma_u(\upsilon_u, d)$, namely, in the region $d \le d_{\max}$, its dependence on υ_u is much greater than its dependence on the grain size.

The boundary conditions of the problem have the form (the SP state region is denoted by the superscript 1, and the ideally plastic state region by the superscript 2)

$$y = h: \quad \sigma_{xy}^{(2)} = -\tau_0 = -\sigma_0/\sqrt{3}, \quad \upsilon^{(2)} = -\upsilon_0 \tag{1.8}$$

$$y = 0: \quad \sigma_{xy}^{(1)} = 0; \quad \upsilon^{(1)} = 0 \tag{1.9}$$

$$y = \eta(t): u^{(1)} = u^{(2)}, \quad \upsilon^{(1)} = \upsilon^{(2)}, \quad \sigma_{xy}^{(1)} = \sigma_{xy}^{(2)}, \quad \sigma_{yy}^{(1)} = \sigma_{yy}^{(2)} \tag{1.10}$$

At the boundary $x = l$, the conditions are satisfied in the integral sense:

$$\int_0^h \sigma_{xx}|_{x=l}dy = 0; \quad \int_0^h u|_{x=l}dy = \frac{\upsilon_0 l}{h} \tag{1.11}$$

In region 2, the stresses satisfy the von Mises plasticity condition.

2. Solution in a first approximation

From the condition of incompressibility we find $u = -(\partial u/\partial y)x + f(y)$, and hence $2u_{xy} = -(\partial^2 \upsilon/\partial y^2)x + \partial f/\partial y$. However, the condition $\upsilon_{xy} = \upsilon_{xy}(y)$ is possible provided $\partial^2 \upsilon/\partial y^2 = 0$; coupled with the conditions (1.8) to (1.10), it follows that

$$\upsilon^{(1)} = \upsilon^{(2)} = \upsilon = -\upsilon_0\frac{y}{h} \tag{2.1}$$

From the equilibrium equations and conditions (1.8) to (1.10), for the stresses we obtain

$$\sigma_{xy}^{(1)} = \sigma_{xy}^{(2)} = \sigma_{xy} = -\frac{\sigma_0}{\sqrt{3}}\frac{y}{h} \tag{2.2}$$

Subsequent analysis for regions 1 and 2 is carried out separately. From the incompressibility equation, taking Eq. (2.1) into account, we obtain

$$u^{(1)} = \frac{\upsilon_0 x}{h} + f^{(1)}(y), \quad u^{(2)} = \frac{\upsilon_0 y}{h} + f^{(2)}(y) \tag{2.3}$$

$$\upsilon_u^{(1)} = \frac{2\upsilon_0}{\sqrt{3}h}(1+\varphi_1'^2)^{1/2}, \quad \upsilon_u^{(2)} = \frac{2\upsilon_0}{\sqrt{3}h}(1+\varphi_2'^2)^{1/2}; \quad \varphi_1 = \frac{hf^{(1)}}{2\upsilon_0}, \quad \varphi_2 = \frac{hf^{(2)}}{2\upsilon_0} \tag{2.4}$$

where a prime (′) denotes a derivative with respect to y.
Comparing the expressions for σ_{xy} from relations (2.2) and the law of plastic flow (1.4), for region 2 we obtain

$$\varphi_2' = -\frac{y}{h}(1+\varphi_2'^2)^{1/2}, \quad \varphi_2 = h\left(1-\frac{y^2}{h^2}\right)^{1/2} + b_2'$$

From Eq. (2.3) we find

$$u^{(2)} = \frac{\upsilon_0 x}{\sqrt{3}h} + 2\upsilon_0\sqrt{1-\frac{y^2}{h^2}} + b_2 \tag{2.5}$$

Now, from the equilibrium equations and the plasticity condition, the stresses are determined

$$\sigma_{xx}^{(2)} = \frac{\sigma_0 x}{\sqrt{3}h} + \frac{2\sigma_0}{\sqrt{3}}\sqrt{1-\frac{y^2}{h^2}} - b, \quad \sigma_{yy}^{(2)} = \frac{\sigma_0 x}{\sqrt{3}h} - b \tag{2.6}$$

In region 1, comparison of the expressions for σ_{xy} from relations (2.2) and (1.4) leads to the equation

$$-\frac{y}{h}(1+\varphi_1'^2)^{1/2} = \varphi_1' F\left(\frac{\upsilon_u^{(1)}}{\upsilon_{\max}}\right) \tag{2.7}$$

Using the first equation in system (2.4), we find

$$-\frac{y}{h}(1+\varphi_1'^2)^{1/2} = \varphi_1' F(\beta(1+\varphi_1'^2)^{1/2}) \equiv \varphi_1'\Phi(\varphi_1'); \quad \beta = \beta(t) = \frac{2\upsilon_0(t)}{\sqrt{3}h(t)\upsilon_{\max}} \tag{2.8}$$

The following important conclusions can be drawn from Eq. (2.8):
(1) $\varphi_1' = 0$ when $y = 0$; $\varphi_1' \le 0$ in the region $0 \le y \le \eta(t)$.
(2) at the boundary of the SP state region $y = \eta(t)$ we have $\upsilon_u^{(1)} = \upsilon_{\max}$, and from Eq. (2.7) we therefore find that

$$\beta(1+\varphi_1'^2(\eta)) = 1, \quad \varphi_1'(\eta) = -(1-\beta^2)^{1/2}/\beta$$

(3) at the same time, $\sigma_u^{(1)} = \sigma_0$ when $y = \eta(t)$; therefore $F(1) = 1$, and then $\eta(t)/h = \sqrt{1-\beta^2}$.
This last result is the most important: the boundary of the SP state region is found, irrespective of the form of the rapid strain hardening function, without requiring complete solution of the problem. It is interesting that, however close to the start of flow ($t \to +0$), the boundary $\eta(t)$ can behave in different ways. We have

$$\eta(0) = h_0\sqrt{1-\beta_0^2}, \quad \beta_0 = 2\upsilon_0(0)/(\sqrt{3}h_0\upsilon_m)$$

where h_0 is the initial layer thickness. If the loading rate υ_0 changes continuously from a zero value, then $\eta(0) = h_0$, i.e., the entire layer is in the SP state; if $\upsilon_0(0) > 0$, however, then two subregions are “instantaneously” formed in the layer: $\eta(0) < h_0$, since $\beta(0) > 0$.

Eq. (2.8) has a unique solution in the region $0 \le y \le \eta(t)$. We will rewrite it in the form

$$-\frac{y}{h} = \frac{\varphi_1'}{(1+\varphi'^2)^{1/2}}\Phi_1(\varphi_1') \equiv \Phi_0(\varphi_1')$$

As Φ_1 is a monotonically increasing continuous function for all materials in the SP state, the function Φ_0 increases monotonically and is continuous. According to a well-known analysis theorem a unique (monotonic and continuous) inverse function $\varphi_1' = \Phi_0^{-1}(y/h) \equiv g_1(y)$ exists, whence $\varphi_1 = g(y) + b_1'$, where $g(y)$ is the primitive of $g_1(y)$.

From the first equation of system (2.3), the velocity $u^{(1)}$ is now determined:

$$u^{(1)} = \frac{\upsilon_0 x}{h} + \frac{2\upsilon_0}{h}g(y) + b_1 \tag{2.9}$$

and, from the equilibrium equations and the plastic flow law (1.4), the stresses

$$\sigma_{xx}^{(1)} = \frac{\sigma_0 x}{\sqrt{3}h} - \frac{2\sigma_0}{\sqrt{3}}\frac{y}{hg_1(y)} - d_1; \quad \sigma_{yy}^{(1)} = \frac{\sigma_0 x}{\sqrt{3}h} - d_1 \tag{2.10}$$

Thus, the solution depends on four constants b, b_1, b_2 and d_1, which are found from conditions (1.8) and (1.9).

3. Examples

In Eq. (2.8) we will put $y/h = y_1$ and $\varphi_1' = -z$ and assume that $\beta\sqrt{1+z^2} = \zeta$; we then obtain

$$\sqrt{\zeta^2 - \beta^2}F(\zeta) = y_1\zeta \tag{3.1}$$

In the first example for the strain hardening function we will assume that $F(\zeta) = \sqrt{\zeta}$; from Eq. (3.1) we find

$$2\zeta = y_1^2 + (y_1^4 + 4\beta^2)^{1/2}$$

from which we have the equation

$$z = -\varphi_1' = \frac{y_1}{\sqrt{2}\beta}\left(y_1^2 + \sqrt{y_1^4 + 4\beta^2}\right)^{1/2} \tag{3.2}$$

We will make the substitution $y^2 = \tau$ in this equation and write it in the form

$$-d\varphi_1 = \frac{1}{2\sqrt{2}\beta h^2}\left(\tau + \sqrt{\tau^2 + 4\beta^2 h^4}\right)^{1/2} d\tau \tag{3.3}$$

After the substitution $\tau + \sqrt{\tau^2 + 4\beta^2 h^2} = w^2$ and integration, and then returning to the former variables, we obtain

$$\varphi_1 = \frac{\sqrt{2}\beta h}{\left(y_1^2 + \sqrt{y_1^4 + 4\beta^2}\right)^{1/2}} - \frac{h}{6\sqrt{2}\beta}\left(y_1^2 + \sqrt{y_1^4 + 4\beta^2}\right)^{3/2} \tag{3.4}$$

Substituting expression (3.2) into the first equation of system (2.10), and substituting expression (3.4) into Eq. (2.9), we find $\sigma_{xx}^{(1)}$ and $u^{(1)}$.

In the second example, we put

$$F(\zeta) = (\alpha + \lambda\zeta)^{1/2}; \quad \alpha = (\sigma_1/\sigma_0)^2, \quad \lambda = 1 - \alpha,$$
$$\sigma_1 = \lim \sigma_u \quad \text{when} \quad \upsilon_u \to 0$$

Eq. (3.1) is reduced to a complete cubic equation

$$\lambda\zeta^3 + (\alpha - y_1^2)\zeta^2 - \lambda\beta^2\zeta - \alpha\beta^2 = 0$$

the general solution of which is written in terms of radicals but is inconvenient for investigation. When α is a small parameter (generalization of the previous example), a unique positive root can be found by iterations:

$$2\lambda\zeta^{(k+1)} = y_1^2 - \alpha + [(y_1^2 - \alpha)^2 + 4\lambda\beta^2(\lambda + \alpha/\zeta^{(k)})]^{1/2}$$

As a zero approximation it is natural to choose the solution from the previous example. The solution is written comparatively simply in a first approximation; subsequent transformations are obvious and are not therefore given.

As a final example we will consider the case of small strain hardening

$$F(\zeta) = 1 - \lambda\varphi(\zeta), \ |\lambda\varphi|^2 \ll 1$$

From Eq. (3.1) we will obtain

$$(\zeta^2 - \beta^2)(1 - \lambda\varphi(\zeta))^2 = y_1^2\zeta^2 \tag{3.5}$$

Representing ζ by an expansion in powers of λ: $\zeta = \zeta_0 + \lambda\zeta_1 + \cdots$, substituting: t into Eq. (3.5), and confirming ourselves to the first two terms of the expansion, we obtain

$$\zeta = \zeta_0 + \lambda\zeta_1 + O(\lambda^2) = \frac{\beta}{\sqrt{1 - y_1^2}}\left(1 + \lambda\frac{y_1^2\varphi(\zeta_0)}{1 - y_1^2}\right) + O(\lambda^2)$$

The successive approximations are written relatively simply for the linear function $\varphi(\varsigma)$. The general case of linear strain hardening reduces Eq. (3.1) to a clear, complete, fourth-degree algebraic equation.

4. Estimation of grain size growth

For a known strain rate ζ, to describe the evolution of the average grain size we can use the corresponding equation; we will adopt it in the form[3] (retaining the notation used earlier[3])

$$\dot{d} = B/(qd^{q-1}) + \lambda\zeta d \tag{4.1}$$

where B, q and λ are certain constants. We will denote the initial grain size by d_0, and we will assume that $\rho = (d/d_0)^q$; the velocity υ_0 will be taken as constant; then

$$h = h_0 - \upsilon_0 t = h_0(1 - \upsilon_0 t/h_0) \equiv h_0\tau$$

We will write the time derivative as a partial derivative (non-linear terms will be omitted); from Eq. (4.1) we obtain

$$\partial\rho/\partial\tau + a_1\zeta\rho = -a_2; \quad a_1 = \lambda h_0/\upsilon_0, \quad a_2 = Bh_0/(\upsilon_0 d_0^q) \tag{4.2}$$

The solution of Eq. (4.2) under the initial conditions $\rho = 1$ when $\tau = 1$ has the form

$$\rho = e^{-a_1\psi(\tau)}\left(1 - a_2\int_1^\tau \exp(a_1\psi(\varsigma))d\zeta\right); \quad \psi(\tau) = \int_0^\tau \zeta(\xi)d\xi \tag{4.3}$$

For the strain rate ζ, we will adopt the exact solution from the first example

$$\zeta(\tau) = \frac{y^2}{2h_0^2\tau^2} + \frac{\beta_0}{\tau^2}\sqrt{\tau^2 + b^2}; \quad b^2 = \frac{3y^4\upsilon_m^2}{16h_0^2\upsilon_0^2}$$

The function $\psi(\tau)$ is easily calculated:

$$\psi(\tau) = -\frac{y^2}{2h_0^2}\frac{1-\tau}{\tau} - \beta_0\left(\frac{\sqrt{\tau^2 + b^2}}{\tau} - \sqrt{1 + b^2} + \ln\frac{1 + \sqrt{1 + b^2}}{\tau + \sqrt{\tau^2 + b^2}}\right)$$

In the region $\tau < 1$ ($t > 0$), as can be seen, $\psi(\tau) < 0$, the integral in parentheses in Eq. (4.3) is negative, and therefore relation (4.3) defines the law of grain size growth.

5. Conclusion

On the basis of the solutions obtained, a theory of flow along the surfaces of a thin layer of SP material (in the isothermal case) can be developed, just like that developed by Il'yushin.[2] An obvious feature of the theory will be that the pressure from the layer on the surface will be determined by the yield point of the material outside the SP state region (see the formulae in system (2.10)). The main attention must be concentrated on the kinematics of the flow, as this determines the structure of the material.

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References

1. Vasin RA, Kiiko IA. Formulation of the initial-boundary-value problem of superplasticity. *Vestn MGU Ser 1 Matematika Mekhanika* 2004;(1):58–61.
2. Il'yushin AA. Problems of the theory of flow of a plastic along surfaces. *Prikl Mat Mekh* 1954;**18**(3):265–88.
3. Vasin RA, Yenikeyev FU. *Introduction to Superplasticity Mechanics.* Part I. Ufa: Gilem; 1998.

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